Noncommutative Tensor Triangular Geometry

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General References

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Foundational works due to Benson-Carlson-Rickard, BIK, BIK-Pevtsova, Hopkins, Neeman, and others.....

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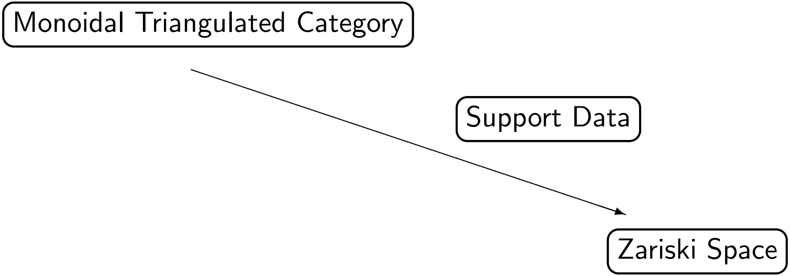
“Categories Have Hidden Geometry”



Monoidal Triangulated Categories

Definition

A *monoidal triangulated category* (MΔC) is a triple $(\mathbf{K}, \otimes, \mathbf{1})$ such that

(i) $\mathbf{K}$ is a triangulated category,

(ii) $\mathbf{K}$ has a monodial tensor product $\otimes : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$ which is exact in each variable with unit object $\mathbf{1}$.

Examples:

Example

Let A be a finite-dimensional Hopf algebra. Then

(iii) $\mathbf{K}^c = \mathrm{stmod}(A)$ stable module category of finite-dimensional modules for A

(iv) $\mathbf{K} = \mathrm{StMod}(A)$ stable module category for $\mathrm{Mod}(A)$.

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Example

Let R be a commutative Noetherian ring. Let

(i) $\mathbf{K}^c = D^b_{perf}(R)$ bounded derived category of finitely generated projective R-modules

(ii) $\mathbf{K} = D(R)$ derived category of R-modules.

Then $\mathbf{K}^c$ and $\mathbf{K}$ are tensor triangulated categories.

“Treat a MΔC Like a Ring”

Definition

(a) A *(tensor) ideal* in $\mathbf{K}$ is a triangulated subcategory $\mathbf{I}$ of $\mathbf{K}$ such that $M \otimes N \in \mathbf{I}$ and $N \otimes M \in \mathbf{I}$ for all $M \in \mathbf{I}$ and $N \in \mathbf{K}$.

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(c) A *completely prime ideal* $\mathbf{P}$ of $\mathbf{K}$ is a proper thick tensor ideal such that if $M \otimes N \in \mathbf{P}$ then either $M \in \mathbf{P}$ or $N \in \mathbf{P}$.

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(d) [NEW] A *prime ideal* $\mathbf{P}$ of $\mathbf{K}$ is a proper thick tensor ideal such that $\mathbf{I} \otimes \mathbf{J} \subseteq \mathbf{P}$ implies that $\mathbf{I} \subseteq \mathbf{P}$ or $\mathbf{J} \subseteq \mathbf{P}$ for all thick ideals $\mathbf{I}$, $\mathbf{J}$ of $\mathbf{K}$. .

A Generalization of Paul Balmer's Categorical Spectrum

Definition

The *Balmer spectrum* is defined as

$$\mathrm{Spc}(\mathbf{K}) = \{\mathbf{P} \subset \mathbf{K} \mid \mathbf{P} \text{ is a prime ideal}\}.$$

The topology on $\mathrm{Spc}(\mathbf{K})$ is given by closed sets of the form

$$Z(\mathcal{C}) = \{\mathbf{P} \in \mathrm{Spc}(\mathbf{K}) \mid \mathcal{C} \cap \mathbf{P} = \varnothing\}$$

where $\mathcal{C}$ is a family of objects in $\mathbf{K}$.

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One can also define

$$\text{CP-}\mathrm{Spc}(\mathbf{K}) = \{\mathbf{P} \subset \mathbf{K} \mid \mathbf{P} \text{ is a completely prime ideal}\}.$$

$$\text{CP-}\mathrm{Spc}(\mathbf{K}) \subseteq \mathrm{Spc}(\mathbf{K}).$$

Zariski Spaces

Definition

Assume throughout that X is a Noetherian topological space. In this case any closed set in X is the union of finitely many irreducible closed sets. We say that X is a *Zariski space* if in addition any irreducible closed set Y of X has a unique generic point (i.e., $y \in Y$ such that $Y = \overline{\{y\}}$).

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Example

Let R is a commutative Noetherian ring.

(1) $X = \text{Spec}(R)$.

(2) $X = \text{Proj}(\text{Spec}(R)) := \text{Proj}(R)$ if R is graded.

(3) $X = G\text{-Proj}(R)$ if R is graded and G is an algebraic group.

Zariski Spaces: Notation

(ii) $\mathcal{X}$ be the collection of subsets of X.

(ii) $\mathcal{X}_{cl}$ be the collection of closed subsets of X.

(iii) $\mathcal{X}_{irr}$ be the set of irreducible closed sets.

(iv) A subset W in X is *specialization closed* if and only if $W = \cup_{j \in J} W_j$ where W_j are closed sets.

(v) $\mathcal{X}_{sp}$ be the collection of all specialization closed subsets of X.

Support Data

Definition

A *support data* (resp. *weak support data*) is an assignment $\sigma : \mathbf{K} \to \mathcal{X}$ which satisfies the following six properties (for $M, M_i, N, Q \in \mathbf{K}$):

(S1) $\sigma(0) = \varnothing$, $\sigma(\mathbf{1}) = X$;

(S2) $\sigma(\oplus_{i \in I} M_i) = \bigcup_{i \in I} \sigma(M_i)$ whenever $\oplus_{i \in I} M_i$ is an object of $\mathbf{K}$;

(S3) $\sigma(\Sigma M) = \sigma(M)$;

(S4) for any distinguished triangle $M \to N \to Q \to \Sigma M$ we have

$$\sigma(N) \subseteq \sigma(M) \cup \sigma(Q);$$

(S5) $\bigcup_{C \in \mathbf{K}} \sigma(M \otimes C \otimes N) = \sigma(M) \cap \sigma(N)$;

(WS5) $\Phi_\sigma(\mathbf{I} \otimes \mathbf{J}) = \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J})$, $\mathbf{I}$ and $\mathbf{J}$ are ideals of $\mathbf{K}$: $\Phi_\sigma(\mathbf{I}) = \cup_{A \in \mathbf{I}} \sigma(A)$.

(S6) $\sigma(M) = \sigma(M^*)$ for $M \in \mathbf{K}^c$ [the compact objects have a duality].

We will be interested in support data which satisfy an additional two properties:

(S7) $\sigma(M) = \varnothing$ if and only if $M = 0$; (Faithfulness Property)

(S8) for any $W \in \mathcal{X}_{cl}$ there exists an $M \in \mathbf{K}^c$ such that $\sigma(M) = W$ (Realization Property).

Classifying Thick Tensor Ideals and the Balmer Spectrum

Theorem (BKN16, Dell'Ambrogio, NVY19)

Let $\mathbf{K}$ *be a compactly generated M*Δ*C. Let* X *be a Zariski space and let* $\sigma : \mathbf{K} \to \mathcal{X}$ *be a weak support data satisfying the additional conditions (S7) and (S8) with* $\sigma(\langle M \rangle) \in \mathcal{X}_{cl}$ *for* $M \in \mathbf{K}^c$*. There is a pair of mutually inverse maps*

$$\{\textit{thick tensor ideals of } \mathbf{K}^c\} \underset{\Theta}{\overset{\Phi_\sigma}{\rightleftarrows}} \mathcal{X}_{sp},$$

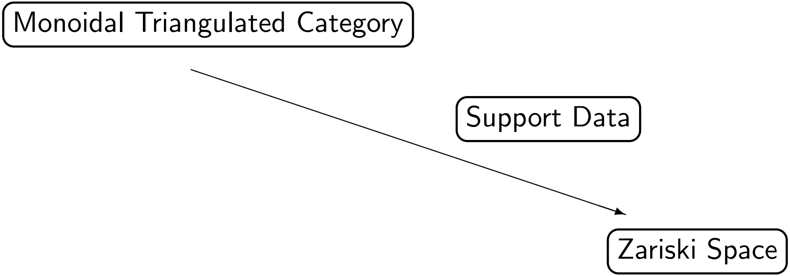
given by

$$\Phi_\sigma(\mathbf{I}) = \bigcup_{M \in \mathbf{I}} \sigma(M), \qquad \Theta(W) = \mathbf{I}_W,$$

where $\mathbf{I}_W = \{M \in \mathbf{K}^c \mid \sigma(M) \subseteq W\}$*. Moreover, there is a homeomorphism*

$$f : X \to \mathrm{Spc}(\mathbf{K}^c).$$

Recap


Monoidal Triangulated Category
Support Data
Zariski Space

Finite Group Schemes [Symmetric MΔC]

Example

Let G be a finite group (scheme), $A := \mathrm{H}^\bullet(G, k) = \mathrm{Ext}^\bullet_G(k, k)$ be the cohomology ring. Set $\mathbf{K}^c = \mathrm{stmod}(G)$ and $X = \mathrm{Proj}(\mathrm{Spec}(A))$.

(i) $\{$thick $\otimes$-ideals of $\mathbf{K}^c\}$ are in one-to-correspondence with $\mathcal{X}_{sp}$.

(ii) $\mathrm{Spc}(\mathbf{K}^c) \cong \mathrm{Proj}(\mathrm{Spec}(A))$.

The (classifying) support data is given by

$$W(M) = \{P \in \mathrm{Proj}(\mathrm{Spec}(A)) : \mathrm{Ext}^\bullet_G(M, M)_P \neq 0\}.$$

Perfect Complexes [Symmetric MΔC]

Example

Let R be a commutative Noetherian ring, $\mathbf{K}^c = D^b_{perf}(R)$ and $X = \mathrm{Spec}(R)$. Then

(i) $\{$thick $\otimes$-ideals of $\mathbf{K}^c\}$ are in one-to-correspondence with $\mathcal{X}_{sp}$.

(ii) $\mathrm{Spc}(\mathbf{K}^c) \cong \mathrm{Spec}(R)$.

The support data which gives this classification is

$$W(C_\bullet) = \{P \in \mathrm{Spec}(R) : \mathrm{H}_*(C_\bullet)_P \neq 0\}.$$

Large Quantum Group [Symmetric MΔC]

Theorem (BKN17)

Let G be a complex simple algebraic group over $\mathbb{C}$ with $\mathfrak{g} = \text{Lie}\, G$. Assume that ζ is a primitive ℓth root of unity where $\ell > h$. Let $\mathbf{K} = \text{Stmod}(U_\zeta(\mathfrak{g}))$, $\mathbf{K}^c = \text{stmod}(U_\zeta(\mathfrak{g}))$ and $X = G\text{-Proj}(\mathbb{C}[\mathcal{N}])$. There exists a 1-1 correspondence

$$\{\textit{thick tensor ideals of } \mathbf{K}^c\} \underset{\Theta}{\overset{\Gamma}{\rightleftarrows}} \mathcal{X}_{sp}.$$

Moreover, there is a homeomorphism

$$\text{Spc}(\text{stmod}(U_\zeta(\mathfrak{g}))) \cong G\text{-Proj}(\mathbb{C}[\mathcal{N}]).$$

Finite-dimensional Hopf-algebras [General MΔC]

Theorem (NVY19)

(a) *Let* $\mathbf{K}$ *be a compactly generated MΔC and* $\sigma : \mathbf{K} \to \mathcal{X}$ *be a quasi support datum for a Zariski space* X *that satisfies the faithfulness and realization properties and the Assumption YY. Then the map*

$$\Phi_\sigma : \{\textit{thick right ideals of } \mathbf{K}^c\} \to \mathcal{X}_{sp}$$

is a bijection.

(b) *Let* A *be a finite-dimensional Hopf algebra over a field* k *that satisfies the standard (Finite Generation) Assumption and the Assumption YY. Set* $X = \mathrm{Proj}(\mathrm{H}^\bullet(A, k))$*. The standard cohomological support* $\sigma : \mathrm{stmod}(A) \to \mathcal{X}_{cl}$ *is a quasi support datum, and as a consequence, there is a bijection*

$$\Phi_\sigma : \{\textit{thick right ideals of } \mathrm{stmod}(A)\} \to \mathcal{X}_{sp},$$

where $\mathrm{stmod}(A)$ *is the stable (finite dimensional) module category of* A*.*

Quantum Groups [Non-commutative MΔC]

Assumptions

Let Δ *be an irreducible root system, and* $\zeta \in \mathbb{C}$ *be a primitive* ℓ*th root of unity.*

- *The integer* ℓ *is odd and greater than* 1*.*
- *If the root system* Φ *has type* G_2*, then* 3 *does not divide* ℓ*.*
- $\ell > h$ *where* h *is the Coxeter number for* Δ*.*

Quantum Groups for Borel Subalgebras

- Δ^+ be the positive roots of an irreducible root system Δ.
- X be the weight lattice
- Γ a $\mathbb{Z}$-lattice with $\mathbb{Z}\Delta \subseteq \Gamma \subseteq X$
- $\{\mu_1, \ldots, \mu_n\}$ a $\mathbb{Z}$-basis for Γ

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Construct $u_{\zeta,\Gamma}(\mathfrak{b}) = u_\zeta(\mathfrak{u})\# u_{\zeta,\Gamma}(\mathfrak{t})$ as follows.

- $u_\zeta(\mathfrak{u})$ is the small quantum group for nilpotent radical of the Borel.
- $u_{\zeta,\Gamma}(\mathfrak{t}) = \mathbb{C}[K_{\mu_1}^{\pm 1}, \ldots, K_{\mu_n}^{\pm 1}]/(K_{\mu_i}^{\ell} - 1, 1 \leq i \leq n)$
- $K_{\mu_i} E_\beta K_{\mu_i}^{-1} = \zeta^{\langle \beta, \mu_i \rangle} E_\beta$.
- The (non-cocommutative) coproduct on $u_\zeta(\mathfrak{u})$ is the standard one used for quantum groups. .

Classification of Thick Tensor Ideals

Theorem (NVY19, NVY20)

Let $u_{\zeta,\Gamma}(\mathfrak{b})$ be the small quantum group for the Borel subalgebra for an arbitrary finite dimensional complex simple Lie algebra. Assume that ℓ satisfies the Assumptions (in particular, $\ell > h$) and $\gcd(\ell, |\Gamma/\mathbb{Z}\Delta|) = 1$, which implies that $R \cong S(\mathfrak{u}^*)$, $X = \mathrm{Proj}(R)$.

(a) *For Φ and Θ as above, there exists a bijection between*

$$\{\textit{thick tensor ideals of } \mathrm{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))\} \underset{\Theta}{\overset{\Phi_W}{\rightleftarrows}} \{\textit{specialization closed sets of } X\}.$$

(b) *Every thick right ideal of $\mathrm{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$ is two-sided.*

(c) *There exists a homeomorphism $f : \mathrm{Proj}(R) \to \mathrm{Spc}(\mathrm{stmod}(u_{\zeta,\Gamma}(\mathfrak{b})))$.*

Methods Used in the Proof

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- A key ingredient involves the work of Benson, Erdmann, and Holloway to relate supports of modules in gr $u_{\zeta,\Gamma}(\mathfrak{b})$ to supports for a quantum complete intersection (with equal parameters) where the theory is better understood via rank varieties.
- By carefully keeping track of the relationship between the support theories between $u_{\zeta,\Gamma}(\mathfrak{b})$ and gr $u_{\zeta,\Gamma}(\mathfrak{b})$ we can successfully verify the key assumption. Much of our analysis uses key facts in the local support theory as developed by Benson, Iyengar, and Krause.

Benson-Witherspoon Hopf Algebras [Non-cocommutative MΔC]

Benson and Witherspoon considered the stable module categories of Hopf algebras of the form

$$A := (k[G]\#kH)^*,$$

where

- G and H are finite groups with H acting on G by group automorphisms,
- k is a field of positive characteristic dividing the order of G,
- kH is the group algebra of H, $k[G]$ is the dual of the group algebra of G,
- A is a non-cocommutative Hopf algebra.

Enlightening Example

Example

Let p be a prime number and n be a positive integer. Benson and Witherspoon analyzed the situation for $G := (\mathbb{Z}/p\mathbb{Z})^n$, $H := \mathbb{Z}/n\mathbb{Z}$ (with H cyclically permuting the factors of G) and k a field of characteristic p,

In this case, A admits a non-projective finite dimensional module M such that $M \otimes M$ is projective. In particular, if W is the cohomological support then

$$W(M \otimes M) \neq W(M) \cap W(M).$$

Classification and the Balmer Spectrum

Theorem (NVY19)

Let $A = (k[G]\#kH)^*$ *where* G *and* H *are finite groups with* H *acting on* G *and* k *is a base field of positive characteristic dividing the order of* G*. Let* $R = \mathrm{H}^\bullet(A, k)$ *and* $X = H\text{-}\mathrm{Proj}(R)$*. The following hold:*

(a) *There exists a bijection*

$$\{\textit{thick tensor ideals of } \mathrm{stmod}(A)\} \underset{\Theta}{\overset{\Phi}{\rightleftarrows}} \{\textit{specialization closed sets of } X\}$$

(b) *There exists a homeomorphism* $f : H\text{-}\mathrm{Proj}(R) \to \mathrm{Spc}(\mathrm{stmod}(A))$*.*

Tensor Product Question

Open Question: When does a support datum $\sigma : \mathbf{K} \to \mathcal{X}_{sp}(X)$ possesses the *tensor product property*

$$\sigma(M \otimes N) = \sigma(M) \cap \sigma(N), \quad \forall M, N \in \mathbf{K}?$$

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- For cohomological supports for modular representations of finite groups (Carlson, Avrunin-Scott) and for finite group schemes (Friedlander-Pevtsova), it is known to hold.
- Many people have been interested in this question for arbitrary finite-dimensional Hopf algebras.

Connection with Completely Prime Ideals I

Theorem (NVY20)

For every monoidal triangulated category $\mathbf{K}$*, the following are equivalent:*

(a) *The universal support datum* $V : \mathbf{K} \to \mathcal{X}(\operatorname{Spc} \mathbf{K})$ *has the tensor product property*

$$V(M \otimes N) = V(M) \cap V(N), \quad \forall M, N \in \mathbf{K}.$$

(b) *Every prime ideal of* $\mathbf{K}$ *is completely prime.*

Universal support datum: $V : \mathbf{K} \to \mathcal{X}$: $V(M) = \{\mathbf{P} \in \operatorname{Spc} \mathbf{K} : \ M \notin \mathbf{P}\}$.

Connection with Completely Prime Ideals II

Theorem (NVY20)

Let $\mathbf{K}$ *be a monoidal triangulated category in which every thick right ideal is two-sided. Then every prime ideal of* $\mathbf{K}$ *is completely prime, and as a consequence, the universal support datum* $V : \mathbf{K} \to \mathcal{X}(\operatorname{Spc} \mathbf{K})$ *has the tensor product property*

$$V(M \otimes N) = V(M) \cap V(N), \quad \forall M, N \in \mathbf{K}.$$

Connections with Nilpotent Elements

Theorem (NVY20)

Let $\mathbf{K}$ *be a monoidal triangulated category in which every object is rigid. If* $\mathbf{K}$ *has a non-zero nilpotent object* M *(i.e.,* $M \ncong 0$ *but* $M^{\otimes n} := M \otimes \cdots \otimes M \cong 0$*, for some* $n > 0$*) then not all prime ideals of* $\mathbf{K}$ *are completely prime. As a consequence, the universal support datum* $V : \mathbf{K} \to \mathcal{X}(\operatorname{Spc} \mathbf{K})$ *does not have the tensor product property.*

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Recall for the Benson-Witherspoon example, there exists a non-zero module M (not projective) such that $M \otimes M = (0)$ in $\mathbf{K}$. Therefore, the universal support datum does not satisfy the tensor product property. This implies that the cohomological (classifying) support datum does not satisfy the tensor product property.

Negron-Pevtsova Conjecture (2020)

Theorem (Nakano-Vashaw-Yakimov, 2020)

Let $u_{\zeta,\Gamma}(\mathfrak{b})$ be the small quantum group for the Borel subalgebra for an arbitrary finite dimensional complex simple Lie algebra. Assume that ℓ satisfies Assumptions on ℓ (in particular, $\ell > h$) and $\gcd(\ell, |\Gamma/\mathbb{Z}\Delta|) = 1$. Then the following hold:

(a) *All prime ideals of $\mathrm{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$ are completely prime.*

(b) *The cohomological support*

$$W(-) : \mathrm{stmod}(u_{\zeta,\Gamma}(\mathfrak{b})) \to \mathcal{X}_{sp}(\mathrm{Proj}(\mathrm{H}^{\bullet}(u_{\zeta,\Gamma}(\mathfrak{b}), \mathbb{C}))$$

has the tensor product property $W(M \otimes N) = W(M) \cap W(N)$ for all $M, N \in \mathrm{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$.

Thank you for your attention.